Calculus of Variations

1. Two fixed points \((x_1, y_1)\) and \((x_2, y_2)\) satisfying \(x_1 < x_2\) are connected by a curve described by the function \(f(x)\). Suppose the curve is revolved around the y-axis to form a surface of revolution. Determine what the function \(f(x)\) must be in order for the surface of revolution to have the smallest possible area.

Hint: The infinitesimal surface area is given by \(dA = 2\pi xdS\).

2. Show that an object sliding down a cycloid reaches the bottom in the same amount of time, regardless of its starting point.

3. Show that if \(L(x, \dot{x})\) is a function of \(x\) and \(\dot{x}\) only and does not explicitly depend on \(t\), then the function \(x(t)\) that extremizes the action satisfies the following equation:

\[ L - \dot{x} \frac{\partial L}{\partial \dot{x}} = \text{constant}. \]

Hint: Try fooling around with \(\frac{d}{dt} \left( \dot{x} \frac{\partial L}{\partial \dot{x}} \right)\) and \(\frac{dL}{dt}\).