Lagrangian Mechanics

1. A block of mass $M_1$ is connected to a spring with spring constant $k_1$ on its left side and a second spring with spring constant $k_2$ on its right side. The other end of the left spring is connected to a wall, and the other end of the right spring is connected to a second block with mass $M_2$ as shown in the figure below. Both blocks rest on a frictionless, horizontal surface. Determine the equations of motion for the two blocks.

2. A small bead with mass $m$ is free to move along a frictionless wire bent into the shape of an ellipse with semi-major axis $b$ (oriented along the vertical) and eccentricity $e$.
   a) Write down a Lagrangian for the particle and determine the equation of motion.
   b) What is the path length for one complete cycle around the ellipse? Express your answer in terms of the elliptic integral of the second kind, $E(\phi, k)$, defined by:

$$E(\phi, k) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta} \, d\theta.$$ 

3. A spring pendulum is formed by replacing the string in a pendulum of mass $m$ with a spring that has a relaxed length $\ell$ and a spring constant $k$. The spring pendulum is then lifted up to an angle $\phi$ clockwise from the negative y-axis, stretched by an amount $\Delta x$, and released.
   a) Determine the equations of motion of the spring pendulum.
   b) Assuming $\Delta x$ and $\phi$ are small, solve for the approximate radial and angular positions as functions of time.
4. A solid sphere of mass $M_B$ and radius $R$ is placed at the top of a triangular wedge of mass $M$ that has an inclination angle $\alpha$. The wedge rests on a frictionless, horizontal surface. When the sphere is released, it begins to roll down the wedge without slipping. Determine the positions of sphere and wedge as functions of time.

5. The end of a simple pendulum with length $\ell$ and mass $m$ is tied to a square peg that is connected to a horizontal track and moved down the positive x-axis as shown in the figure below. The position of the peg as a function of time is $x_p(t) = \frac{1}{6}at^3$.

a) Determine the equation of motion for the pendulum.

b) Assuming a small angle of oscillation, solve the equation of motion to determine the pendulum’s position as a function of time. Comment on the limitations of your answer.