The Wavefunction

1. The wavefunction for a particle at some point in time is given by

\[ \psi(x) = \sqrt{\frac{a}{\pi(x^2 + b^2)}}, \]

where \( a \) and \( b \) are positive, real constants. If the position of the particle is measured at this point in time, what is the probability that it will be found in the range \(-b < x < b\)?

2. At time \( t = 0 \) a particle is represented by the wavefunction

\[ \Psi(x, 0) = \begin{cases} 
A \left( \frac{x}{a} \right), & 0 \leq x \leq a \\
A \left( \frac{b-x}{b-a} \right), & a < x \leq b \\
0, & \text{elsewhere}
\end{cases} \]

a) Normalize and sketch the wavefunction \( \Psi(x, 0) \).

b) Where is the particle most likely to be found at \( t = 0 \)?

c) What is the probability of finding the particle to the right of point \( a \)?

d) Determine the expectation values of \( x \) and \( x^2 \).

3. At time \( t = 0 \) the wavefunction for a particle that is confined to the region \( 0 \leq x \leq \frac{2\pi}{a} \) is given by

\[ \Psi(x, 0) = Ax \left( x - \frac{\pi}{a} \right) \sin(ax), \]

where \( A \) and \( a \) are real constants.

a) What are the units of \( A \) and \( a \)?

b) Normalize and sketch the wavefunction \( \Psi(x, 0) \). How many nodes does the wavefunction have?

c) Calculate \( \langle x \rangle \), \( \langle x^2 \rangle \), and \( \sigma_x \) at \( t = 0 \).

d) What is the probability of finding the particle between the first two nodes?
4. Normalize the following wavefunction and calculate $⟨x⟩$, $⟨x^2⟩$, and $σ_x$

\[ \Psi(x,t) = A e^{-\lambda|x|-i\omega t}. \]

5. At time $t = 0$ a particle is represented by the wavefunction

\[ \Psi(x,0) = \begin{cases} 
A(a^2 - x^2), & -a \leq x \leq a \\
0, & \text{elsewhere.} 
\end{cases} \]

a) Normalize and sketch the wavefunction $Ψ(x,0)$.
b) Calculate $⟨x⟩$ and $⟨x^2⟩$ at $t = 0$.
c) Calculate $⟨p⟩$ and $⟨p^2⟩$ at $t = 0$.
d) Determine the uncertainties in $x$ and $p$ (i.e. calculate $σ_x$ and $σ_p$).
e) Is the result consistent with the uncertainty principle?

The Schrödinger Equation

6. Use the time-dependant Schrödinger equation to prove the following relationship:

\[ \frac{d⟨p⟩}{dt} = \leftlangle -\frac{\partial V}{\partial x} \right⟩. \]

What is the significance of this result?

Probability

7. A needle of length $ℓ$ is dropped at random on a floor uniformly tiled by rectangles whose sides have length $a$ and width $b$. Determine the probability that the needle will land entirely within one of the rectangles so that it does not overlap the boundaries of any tile. You may assume that $ℓ < a, b$. 