Formalism & Bra-ket Notation

1. Calculate the eigenvalues and normalized eigenvectors of the matrix $A$ given below. This should be done by hand, not using Mathematica or other computational software.

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 8 \end{pmatrix}$$

2. Using bra-ket notation, prove that for finite-dimensional vector spaces, any two operators $A$ and $B$ satisfy $\text{Tr}(AB) = \text{Tr}(BA)$.

3. Suppose the Hamiltonian for a given two-state system is

$$H = E_0 \left( |1\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 1| - |2\rangle\langle 2| \right).$$

   a) Write down the matrix representing the Hamiltonian when we represent the kets $|1\rangle$ and $|2\rangle$ by the two unit vectors $(1, 0)$ and $(0, 1)$ respectively.

   b) Determine the normalized energy eigenkets for the above Hamiltonian.

   c) If the system is in the state $|\psi\rangle = |1\rangle$, what are the possible results of an energy measurement and their corresponding probabilities?

   d) Determine the energy expectation value $\langle E \rangle$ for the state $|\psi\rangle = |1\rangle$. 