**Ordinary Quantum Mechanics**

- Physical systems are represented by state vectors in an abstract vector space.

Vector spaces... (kets!) \( |\alpha\rangle + |\beta\rangle = |\gamma\rangle \)

\[ \uparrow \quad \uparrow \quad \uparrow \]
\[ \text{Ket} + \text{Ket} = \text{Some Other Ket} \]

*If you multiply a state vector by a complex number, it still represents the same state!* \( |\alpha\rangle \sim a |\alpha\rangle \)

- Observables are represented by operators...

Operators act on kets from the left: \( A |\alpha\rangle = |\beta\rangle \)

\[ \uparrow \]
\[ \text{Special kets s.t. } A |a_i\rangle = a_i |a_i\rangle \]

\[ \text{Eigenkets!} \]

*The } a_i *are the eigenvalues...
**Example: Spin-\( \frac{1}{2} \) System**

\[
S_z |+\rangle = \frac{1}{2} |+\rangle \\
S_z |\rangle = -\frac{1}{2} |\rangle
\]

*The operator \( S_z \) represents the z-component of spin. The eigenkets of \( S_z \) are \(|+\rangle \& |\rangle \) and the eigenvalues are \( \pm \frac{1}{2} \).*

**Notice that the dimension of our space is determined by the number of possible outcomes of an observation (in this case, two: spin up & spin down).**

**BRA Space**

**For every vector space \( V \) there exists a dual space \( V^* \). There is a 1-1 correspondence between elements in \( V \) and \( V^* \).**

\[
|\alpha\rangle \in V \quad \leftrightarrow \quad \langle\alpha| \in V^*
\]

Ket Space \quad Dual Space (BRA Space)
Let \( \mathcal{D} \) be the map that sends each ket to its corresponding bra: \( \mathcal{D}(|\alpha\rangle) = \langle \alpha | \),

then:

\[
\mathcal{D}(a|\alpha\rangle) = a^* \langle \alpha | \not= a \langle \alpha |
\]

Complex conjugate!

When a bra hits the left of a ket... the whole thing becomes a complex number. This is called the inner product. \( |\alpha\rangle \in \mathcal{V} \) \( \langle \beta | \in \mathcal{V}^* \)

\[
\Rightarrow \langle \beta | \alpha \rangle \in \mathbb{C}
\]

**Axiom:** \( \langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle^* \)

\( \Rightarrow \) this implies \( \langle \alpha | \alpha \rangle \in \mathbb{R} ! \)

**Axiom:** \( \langle \alpha | \alpha \rangle \geq 0 \) (with = only if \( |\alpha\rangle \) is the zero ket)

Note: this lets us normalize any ket... \( |\alpha\rangle \rightarrow \frac{1}{\sqrt{\langle \alpha | \alpha \rangle}} |\alpha\rangle \)
Recall that the kets $|x\rangle$ and $a|\langle x|y\rangle$ represent the same state. This means we are always free to normalize our state vector!

* Two kets $|x\rangle$ and $|y\rangle$ are orthogonal if $\langle x|y\rangle = 0$

Operators act on kets from the left, but act on bras from the right! $A|x\rangle \langle y|B$ → $\langle y|B\langle x|A|x\rangle$

Properties of operators...

1) If $A|x\rangle = B|x\rangle \ \forall \ |x\rangle \in V \Rightarrow A = B$

2) $A|x\rangle = 0 \ \forall \ |x\rangle \in V \Rightarrow A = 0 \ \ (\text{zero operator})$

3) $A + B = B + A$

4) $A + (B + C) = (A + B) + C$

Lastly, $\text{D}(A|x\rangle) = \langle x|A^+ \rightarrow A^+$ is called the adjoint of the operator $A$. This is the definition of adjoint!
→ An operator is Hermitian if \( A^+ = A \) (self adjoint).

Note: Operator multiplication is associative, but not commutative! \( A(BC) = (AB)C \) but \( AB \neq BA \)

Q: What is \((AB)^+\)?

Well... \( AB |\alpha\rangle = A (B |\alpha\rangle) = A |\beta\rangle \)

So... \( \langle \alpha | (AB)^+ = D(A |\beta\rangle) = \langle \beta | A^+ \)

But \( \langle \beta | = D(B |\alpha\rangle) = \langle \alpha | B^+ \)

Hence, \( \langle \alpha | (AB)^+ = \langle \alpha | B^+ A^+ \Rightarrow (AB)^+ = B^+ A^+ \).

Outer product # What if we had something like \(|\alpha\rangle \langle \beta|\)? What kind of object is this?

... What happens if it hits a ket?
IF IT HITS A KET FROM THE LEFT IT GIVES YOU A NEW KET! IF IT HITS A BRA FROM THE RIGHT IT GIVES YOU A NEW BRA!

\[ |x \rangle \langle y | \] IS AN OPERATOR!

\[ (|x \rangle \langle y |) |x \rangle = \langle y | x \rangle |x \rangle \leftarrow \text{NEW KET} \]

\[ \uparrow \text{KET COMPLEX NUMBER} \]

\[ \langle x | (|x \rangle \langle y |) = \langle x | x \rangle \langle y | \leftarrow \text{NEW BRA} \]

\[ \uparrow \text{BRA COMPLEX NUMBER} \]

HENCE:

\[ |x \rangle \text{ KET} \quad |x \rangle \langle y | \text{ OPERATOR} \]

\[ \langle x | \text{ BRA} \quad \langle x | y \rangle \text{ NUMBER} \]

But! \[ A |x \rangle, \quad B \langle y |, \quad \langle x | \langle y | |x \rangle |y \rangle \leftarrow \text{NONSENSE!!} \]

\[ \uparrow \text{"ILLEGAL PRODUCTS"} \]

THEOREM: THE EIGENVALUES OF A HERMITIAN OPERATOR ARE REAL, AND THE EIGENKETS CORRESPONDING TO DISTINCT EIGENVALUES ARE ORTHOGONAL.
Proof: Let \( |\alpha_i\rangle \) be the eigenkets of \( A = A^* \).

\[
A |\alpha_i\rangle = a_i |\alpha_i\rangle \implies \langle a_i | A^* = a_i^* \langle a_i |
\]

But \( A^* = A \) so...

\[
\langle a_i | A^* = \langle a_i | A = a_i^* \langle a_i |
\]

Now multiply both sides on the right by \( |\alpha_i\rangle \)

\[
= \langle a_i | A |\alpha_i\rangle = a_i^* \langle a_i | a_i\rangle
\]

\[
= a_i^* |\alpha_i\rangle
\]

Hence,

\[
a_i^* \langle a_i | a_i\rangle = a_i^* \langle a_i | a_i\rangle
\]

\[
= \langle a_i | a_i\rangle (a_i - a_i^*) = 0
\]

If \( i = j \) then \( a_i = a_i^* \implies a_i \text{ is real!} \)

If \( i \neq j \) then \( a_i - a_j \neq 0 \implies \langle a_i | a_j \rangle = 0! \)

\[\text{Q.E.D.}\]
Since we can normalize any ket, this means the eigenkets of such a Hermitian operator form an orthonormal set! If they span the space, it is a complete set...

But recall that the dimension of the space is determined, in many cases, by the number of possible outcomes when measuring an observable...

If the eigenkets $|a_i\rangle$ form a complete set then...

→ For any ket in $V$

$|\psi\rangle = \sum_{i=1}^{N} c_i |a_i\rangle$ and $\langle a_i | a_j \rangle = \delta_{ij}$. 

So...

$\langle a_i | \psi \rangle = \sum_{j=1}^{N} c_j \langle a_i | a_j \rangle = C_i$

$\Rightarrow$ $|\psi\rangle = \sum_{i=1}^{N} \langle a_i | \psi \rangle |a_i\rangle = \sum_{i=1}^{N} |a_i\rangle \langle a_i | \psi \rangle$
Hence, \[
\left( \sum_{i=1}^{N} |a_i \rangle \langle a_i| \right) \mid \psi \rangle = \mid \psi \rangle \Rightarrow \sum_{i=1}^{N} |a_i \rangle \langle a_i| = I
\]

AN OPERATOR

Completeness Relation

THIS IS A USEFUL THING!!!

Even though it seems trivial...

Compare to:

\[
\left( \hat{V}_x \right)^2 + \left( \hat{V}_y \right)^2 + \left( \hat{V}_z \right)^2 = \hat{V}
\]

\[
\langle a_i | \psi \rangle = \text{"the amount of } \mid \psi \rangle \text{ in the } \mid a_i \rangle \text{ direction"}
\]

\[
\| C_i \text{ \rightarrow expansion coefficient}
\]

Observables are represented by hermitian operators! So... the eigenvalues are real, and the eigenkets form a complete, orthonormal set. Any ket in the space can be expressed as a linear combination of the eigenkets.

\[
\mid \psi \rangle = \sum_{\hat{a}=1}^{N} <a_i | \psi \rangle \mid a_i \rangle
\]
The physical system is represented by some state ket $|\psi\rangle$.

**Rules of Quantum Mechanics**

1. When a measurement is made, the result is one of the eigenvalues of the operator representing the observable being measured.

2. Whatever state the system was in prior to being measured, it disregards and instead suddenly "jumps" into the eigenket corresponding to the eigenvalue that the measurement produced. This changes the state! (Unless the state was already an eigenket.)

3. The probability that a measurement of the system in state $|\psi\rangle$ will yield eigenvalue $\alpha$ is equal to $|\langle \alpha | \psi \rangle|^2$. The state is then $|\alpha\rangle$. 