Stern-Gerlach Experiment

- A beam of silver atoms is sent through a magnetic field and split according to the atoms' angular momenta.

- Let the direction of the $\mathbf{B}$-field be the $\mathbf{z}$-direction. Now this apparatus is measuring the z-component of each Ag atom.

Q: What do we expect on the detector?

Classical

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<th>Actual</th>
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All values $|\vec{\mathbf{\hat{z}}} |\leq \mu_z \leq |\vec{\mathbf{\hat{z}}} |

50% 50%

$\uparrow$ only two values!

Since $\hat{N} \propto \hat{S}_z$, what we learn is that the beam is split into only two parts: those having $S_z = \frac{1}{2}$ and those having $S_z = -\frac{1}{2}$. No other $S_z$ values show up.
We can prepare multiple Stern-Gerlach apparatuses in different directions and run them in succession!

\[ S_Z^+ \rightarrow S_Z + (100\%) \]

Block this beam

\[ S_Z^+ \rightarrow S_x + (50\%) \rightarrow S_x - (50\%) \]

Blocked off again

\[ S_Z^+ \rightarrow S_x + (50\%) \rightarrow S_z - (50\%) \]

Still blocked

Also block off \( S_x^- \) beam

The \( S_z^- \) beam has been resurrected!

Q: How can we describe this using the math we have developed?

A: First, the system is described by a state \(|\psi\rangle\) and there are operators: \( S_z, S_x, \) and \( S_y \).
THE OPERATORS $S_z, S_x,$ & $S_y$ EACH HAVE ONLY TWO EIGENVALUES ($\pm \frac{1}{2}$) AND THE TWO CORRESPONDING EIGENKETS: $|S_z, +\rangle$ AND $|S_z, -\rangle$ FOR $S_z$.

REMEMBER THAT THE DIMENSION OF THE VECTOR SPACE IS DETERMINED BY THE NUMBER OF POSSIBLE OUTCOMES. FOR US, THIS IS TWO (SPIN UP OR SPIN DOWN).

THE $S_z \uparrow$ BEAM IS IN THE $|S_z, +\rangle$ EIGENKET BECAUSE MEASURING THE Z-COMPONENT OF ITS SPIN ALWAYS YIELDS SPIN UP.

Any state in this 2-dimensional space can be expressed as a linear combination of $|S_z, +\rangle$ and $|S_z, -\rangle$!

$$|\psi\rangle = a |S_z, +\rangle + b |S_z, -\rangle$$

WHAT IS THE OPERATOR $S_z$? WELL... WE KNOW WHAT IT DOES TO THE EIGENKETS, WHICH ACT AS BASIS VECTORS FOR OUR STATE...
\[ S_z |S_z, +\rangle = \frac{\hbar}{2} |S_z, +\rangle \]
\[ S_z |S_z, -\rangle = -\frac{\hbar}{2} |S_z, -\rangle \]

... and remember that we can create operators using products like \(|x\rangle \langle y|\)...

So...

\[ \Rightarrow S_z = \frac{\hbar}{2} \left[ |S_z, +\rangle \langle S_z, +| - |S_z, -\rangle \langle S_z, -| \right] \]

This is what \(S_z\) is!

**CHECK:**

\[ S_z |\psi\rangle = S_z (a |S_z, +\rangle + b |S_z, -\rangle) \]

\[ \text{Arbitrary Ket} = a S_z |S_z, +\rangle + b S_z |S_z, -\rangle = \frac{\hbar}{2} \left\{ a |S_z, +\rangle - b |S_z, -\rangle \right\} \]

\[ \frac{\hbar}{2} \left[ |S_z, +\rangle \langle S_z, +| - |S_z, -\rangle \langle S_z, -| \right] \left( a |S_z, +\rangle + b |S_z, -\rangle \right) \]

\[ = \frac{\hbar}{2} \left\{ a |S_z, +\rangle - b |S_z, -\rangle \right\} \text{ SAME THING!} \]

And remember... if \( A |\psi\rangle = B |\psi\rangle \) for arbitrary \(|\psi\rangle\), then \( A = B \)!

What about \(|S_x, +\rangle\), \(|S_x, -\rangle\) and \(S_x\)?
Since any vector can be written as a linear combination of $|S_z, +>$ and $|S_z, ->$ then the $S_x$ eigenket $|S_x, +>$ can as well:

$$|S_x, +> = a|S_z, +> + b|S_z, ->$$

And since a $S_x^+$ beam splits into 50% $S_z^+$ and 50% $S_z^-$ when subject to a $SG^2$ apparatus (3rd picture a few pages back), we know:

$$|\langle S_z, +|S_x, +\rangle|^2 = \frac{1}{2} \quad \text{and} \quad |\langle S_z, -|S_x, +\rangle|^2 = \frac{1}{2}$$

Hence $a = \frac{1}{\sqrt{2}}$ and $b = \frac{1}{\sqrt{2}} e^{i\theta}$

$$\Rightarrow |S_x, +> = \frac{1}{\sqrt{2}} |S_z, +> + \frac{1}{\sqrt{2}} e^{i\theta} |S_z, ->$$

Q: What's with the $e^{i\theta}$?

A: Well, we only know that the absolute values of $\langle S_z, +|S_x, +\rangle$ and $\langle S_z, -|S_x, +\rangle$ are $\frac{1}{\sqrt{2}}$, so there could be a complex phase. We don't have one in front of $|S_z, +>$ because we can pull out...
An overall phase in front of the whole thing and it won't matter, so why not make it equal to whatever phase is in front of $|S_z, +\rangle$...

\[ \text{Ex: } e^{i\theta} |S_z, +\rangle + e^{i\theta_2} |S_z, -\rangle = e^{i\theta} \left( |S_z, +\rangle + e^{i(\theta_2 - \theta)} |S_z, -\rangle \right) = e^{i\theta} \]

**OVERALL PHASE DOES NOT MATTER**

Now! Since $|S_x, +\rangle$ is orthogonal to $|S_x, -\rangle$ we have:

\[ |S_x, -\rangle = \frac{1}{\sqrt{2}} |S_z, +\rangle - \frac{1}{\sqrt{2}} e^{i\theta} |S_z, -\rangle \]

\[ \uparrow \text{EVERYTHING WE DID CAN BE REPEATED FOR } S_y \ldots \]

\[ \Rightarrow |S_y, \pm\rangle = \frac{1}{\sqrt{2}} |S_z, +\rangle \pm \frac{1}{\sqrt{2}} e^{i\phi} |S_z, -\rangle \]

\[ \uparrow \text{DIFFERENT FROM THE PHASE IN } |S_x, \pm\rangle \]

Q: Can we figure out the phase factors?

A: We need one more SG experiment...
\[ S_x^+ \rightarrow S_y^+ (S_0 \%) \]
\[ \uparrow \text{Block the } S_x^- \text{ BEAM} \]
\[ S_y^- (S_0 \%) \]

Hence \[ |S_y^+, S_x^+\rangle = |S_y^-, S_x^+\rangle = \frac{1}{\sqrt{2}} \]

This tells us that \( \theta - \phi = \pm \frac{\pi}{2} \)

**Conclusion:** The spin eigenkets cannot all be real! You must have \( |S_x, \pm\rangle \) or \( |S_y, \pm\rangle \) have imaginary parts!

**Final Result:**
\[ |S_x, \pm\rangle = \frac{1}{\sqrt{2}} |S_z, +\rangle \pm \frac{1}{\sqrt{2}} |S_z, -\rangle \]
\[ |S_y, \pm\rangle = \frac{1}{\sqrt{2}} i |S_z, +\rangle \pm \frac{1}{\sqrt{2}} |S_z, -\rangle \]

And:
\[ S_x = \frac{\hbar}{2} \left[ |S_z, +\rangle \langle S_z, -| + |S_z, -\rangle \langle S_z, +| \right] \]
\[ S_y = \frac{\hbar}{2} \left[ i |S_z, +\rangle \langle S_z, -| - i |S_z, -\rangle \langle S_z, +| \right] \]
\[ S_z = \frac{\hbar}{2} \left[ |S_z, +\rangle \langle S_z, +| - |S_z, -\rangle \langle S_z, -| \right] \]

Wow! Now we know everything!