CONTINUOUS SPECTRA

Q: WHAT IF AN OBSERVABLE HAS CONTINUOUSLY MANY POSSIBLE OUTCOMES?

A: THEN THE EIGENVALUES FORM A CONTINUOUS SPECTRUM, AND THE VECTOR SPACE IN WHICH THE SYSTEM'S STATE KET LIVES IS AN INFINITE DIMENSIONAL HILBERT SPACE.

EX: THE POSITION OPERATOR: $\hat{X}$ (HATS INDICATE OPERATOR)

LET $|x>$ BE AN EIGENKET OF $\hat{X}$

$\Rightarrow \quad \hat{X}|x> = x|x>$

OPERATOR NUMBER (EIGENVALUE)

FOR CONTINUOUS SPECTRA:

1) $\langle a_\alpha | a_\beta^* \rangle = \delta_{\alpha\beta} \rightarrow \langle x | x' \rangle = \delta(x-x')$

2) $\sum_{\beta=1}^{N} |a_\beta \rangle \langle a_\beta | = | \rightarrow \int_{-\infty}^{\infty} dx' |x'\rangle \langle x' | = |$
Let's expand an arbitrary state $|\psi\rangle$ in terms of the position eigenkets (remember that they form a complete set — the mathematics of infinite dimensional Hilbert spaces is perilous, so we'll take this as a postulate of our theory).

$$|\psi\rangle = \int_{-\infty}^{\infty} dx' |x'\rangle \langle x'| \psi\rangle = \int_{-\infty}^{\infty} \langle x'| \psi\rangle |x'\rangle dx'$$

$\langle x'| \psi\rangle = $ a number that depends on $x'$.

**Normalization:** $\langle \psi | \psi \rangle = 1$

Hence,

$$\langle \psi | \psi \rangle = \langle \psi | \int_{-\infty}^{\infty} dx' |x'\rangle \langle x'| \psi\rangle$$

$$= \int_{-\infty}^{\infty} \langle \psi | x'\rangle \langle x' | \psi\rangle dx'$$

But recall that $\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle^*$ so...

$$\langle \psi | \psi \rangle = \int_{-\infty}^{\infty} |\langle x'| \psi\rangle|^2 dx' = 1.$$
* Does this look familiar?

* What do you call a number that depends on a variable?

  \[ \rightarrow A: \text{a function...} \]

- There is a special name for the inner product \( \langle x | \psi \rangle \), it's called the wavefunction!

\[ \langle x | \psi \rangle = \psi(x) \]

The wavefunction is not the state (the state is a vector), the wavefunction is the inner product of the state with an arbitrary position eigenket.

Note: You can think of functions as basically representing infinite dimensional vectors.

Can approximate

\[ \vdash = (v_1, v_2, v_3, \ldots, v_n) \quad \text{N-D} \]

\[ \vdash = (v(x)) \quad \text{\&-D} \]

\[ \vdash \uparrow \]

label

components w/ continuous index
Q: Once you know what the state is, what happens to it after that? Assuming no measurements are made, how does the system change? Does it just stay the same?

A: No! The system, if left alone, does change... but how?

Let $\mathcal{N}(t,t_0)$ be the time evolution operator

$$\mathcal{N}(t,t_0) |\psi, t_0\rangle = |\psi, t\rangle$$

What is the form of $\mathcal{N}(t,t_0)$? Well... we know it must satisfy several properties:

1. $\mathcal{N}(t,t_0) + \mathcal{N}(t,t_0) = 1$ (Unitarity)

Since the state must remain normalized, we have:

$$\langle \psi, t | \psi, t \rangle = \langle \psi, t_0 | \mathcal{N}(t,t_0)^+ \mathcal{N}(t,t_0) | \psi, t_0 \rangle = 1$$

$$= 1 \quad \text{a/c} \quad \langle \psi, t_0 | \psi, t_0 \rangle = 1$$

2. $\lim_{\Delta t \to 0} \mathcal{N}(t+\Delta t, t) = 1$
(3) \( \mathcal{N}(t_2, t_1) \mathcal{N}(t_1, t_0) = \mathcal{N}(t_2, t_0) \) \hspace{1cm} \text{TRANSITIVITY OF TIME EVOLUTION!}

Now, for an infinitesimal time evolution \( dt \), we may write:

\[ \mathcal{N}(t+dt, t) = 1 - i \Omega dt \]

for some Hermitian operator \( \Omega \) (i.e., \( \Omega^\dagger = \Omega \)) since all of these conditions are satisfied:

(1) \( \mathcal{N}^\dagger \mathcal{N} = (1 + i \Omega^\dagger dt)(1 - i \Omega dt) = 1 - i \Omega^\dagger dt + i \Omega dt + \Omega^2 dt^2 \)

\[ \Rightarrow \mathcal{N}^\dagger \mathcal{N} = 1 + \mathcal{O}(dt^2) \]

(2) \( \lim_{dt \to 0} (1 - i \Omega dt) = 1 \)

(3) \( \mathcal{N}(t+2dt, t+dt) \mathcal{N}(t+dt, t) = (1 - i \Omega dt)(1 - i \Omega dt) \)

\[ = 1 - i \Omega (2dt) + \mathcal{O}(dt^2) \]

\[ = \mathcal{N}(t+2dt, t) \]

\[ \text{OK, BUT WHAT IS } \Omega? \]

\[ \text{WE NEED SOME HERMITIAN OPERATOR WITH UNITS OF FREQUENCY... WE'LL RECALL THE ENERGY RELATION } E = \hbar \omega \text{ WHICH MOTIVATES US TO TRY} \]

\[ \Omega = \frac{H}{\hbar} \text{ WHERE } H \text{ IS THE HAMILTONIAN OPERATOR.} \]
It turns out empirically that this is the correct choice!

\[ \Rightarrow \quad \mathcal{H}(t+dt, t) = 1 - \frac{i\hbar}{\hbar} dt \]

Q: How can we find \( \mathcal{H}(t, t_0) \) for finite time \( t \)?

A: Let’s use property \#3: \( \mathcal{H}(t_2, t_1) \mathcal{H}(t_1, t_2) = \mathcal{H}(t_2, t_0) \)

with \( t_2 = t+dt, \ t_1 = t, \) and \( t_0 = t_0 \).

\[ \mathcal{H}(t+dt, t_0) = \mathcal{H}(t+dt, t) \mathcal{H}(t, t_0) \]

\[ = (1 - \frac{i\hbar}{\hbar} dt) \mathcal{H}(t, t_0) \]

\[ = \mathcal{H}(t, t_0) - \frac{i\hbar}{\hbar} dt \mathcal{H}(t, t_0) \]

\[ \Rightarrow \]

\[ \lim_{dt \to 0} \frac{\mathcal{H}(t+dt, t_0) - \mathcal{H}(t, t_0)}{dt} = -\frac{i\hbar}{\hbar} \mathcal{H}(t, t_0) \]

A differential equation for the operator \( \mathcal{H} \)!

Now multiply both sides by the ket \( |\chi, t_0\rangle \) on the right:

\[ \frac{i\hbar}{\hbar} \frac{\partial}{\partial t} \mathcal{H}(t, t_0) |\chi, t_0\rangle = \mathcal{H}(t, t_0) |\chi, t_0\rangle \]
But $\Psi(t, x_0) \left| x, t_0 \right> = \left| x, t \right>$ so this yields:

$$i\hbar \frac{\partial}{\partial t} \left| x, t \right> = H \left| x, t \right>$$

This is the Schrödinger equation!

To get it in terms of the wave function, just take the inner product with a position eigenket $\left| x \right>$.

$$\left< x \left| \frac{\partial}{\partial t} \right| \Psi(t) \right> = \left< x \left| H \right| \Psi(t) \right>$$

Hamiltonian operator acting on state ket

$$\implies i\hbar \frac{\partial}{\partial t} \left< x \left| \Psi(t) \right> = \hat{H} \left< x \left| \Psi(t) \right> \right.$$ 

Differential operator acting on inner product $\left< x \left| \Psi(t) \right> \right.$

So...

$$i\hbar \frac{\partial}{\partial t} = H \Psi(x,t)$$

Schrödinger equation!

Let's find a solution for $\Psi(t, x_0)$ assuming the Hamiltonian is not time dependent...

- We can build $\Psi(t, x_0)$ out of successive infinitesimal time evolutions...

$$\Psi(t, x_0) = \lim_{n \to 0} \Psi(t + \Delta t, x_0)$$
To make this precise: 
\[ \mathcal{N}(t, t_0) = \lim_{N \to \infty} \left( 1 - \frac{\tilde{H}(t-t_0)}{\hbar} \right)^N \]
\[ = e^{-\frac{\Delta E(t-t_0)}{\hbar}} \]

Hence
\[ \mathcal{N}(t, t_0) = e^{-\frac{\Delta E(t-t_0)}{\hbar}} \]

Since \( H|\text{E}_N\rangle = \text{E}_N |\text{E}_N\rangle \), we know \( \mathcal{N}(t, t_0) |\text{E}_N\rangle = e^{-\frac{\Delta \text{E}_N (t-t_0)}{\hbar}} |\text{E}_N\rangle \)

So we know how energy eigenkets evolve in time! They just pick up a complex phase factor!

Q: How does an arbitrary state evolve in time?

\[ |\psi\rangle = \sum_{N=1}^{\infty} \langle \text{E}_N | \psi \rangle |\text{E}_N\rangle \]

\[ \Rightarrow |\psi, t\rangle = \sum_{N=1}^{\infty} \langle \text{E}_N | \psi \rangle e^{-\frac{i \text{E}_N t}{\hbar}} |\text{E}_N\rangle \]

Just expand the state in terms of energy eigenkets and apply the time evolution operator!