Linear Algebra

1. Do the vectors \((2, 0, 1), (1, 3, 2),\) and \((0, 4, 6)\) form a basis for \(\mathbb{R}^3\)? If so, prove it by showing that they are linearly independent and that any vector in \(\mathbb{R}^3\) can be expressed as a linear combination of them. If not, explain why they don’t.

2. Show that the set of polynomials \(\{f_1(x), f_2(x), f_3(x), f_4(x)\}\) is linearly dependent by expressing \(f_4(x)\) as a linear combination of the others.

\[
\begin{align*}
  f_1(x) &= x - 3x^2 + x^5 \\
  f_2(x) &= 2x^2 - x^3 + 6x^4 \\
  f_3(x) &= 1 + 5x - 2x^3 + 3x^5 \\
  f_4(x) &= 2 + 4x + 5x^3 - 54x^4
\end{align*}
\]

3. Suppose \(V\) is the vector space consisting of all \(n \times n\) complex matrices, and \(A\) is some fixed matrix in that space. Let \(T_A : V \rightarrow V\) be the map defined by \(T_A(X) = AXA^\dagger\). Determine whether \(T_A\) is or is not a linear transformation and prove your conclusion.

4. It is possible to think of complex numbers as linear transformations on a two-dimensional, real vector space. Suppose we identify each two-dimensional vector \(\vec{v} = (x, y)\) with the corresponding number in the complex plane, \(\vec{v} \sim v \equiv x + iy\) in this case. We can then define an arbitrary complex number \(z\) as the linear transformation that maps the vector \(\vec{v}\) to the vector corresponding to the complex number \(zv\). Determine the matrix representing an arbitrary such complex number and write down the corresponding basis for the complex plane. Using your results, consider only unit length complex numbers and determine how their matrix representations can be most conveniently parameterized. Consider the result and comment.

5. Calculate the eigenvalues and eigenvectors of the matrix \(A\) given below (you do not need to normalize the eigenvectors for this problem). This should be done by hand, not using Mathematica or other computational software.

\[
A = \begin{pmatrix}
  3 & 5 & 8 \\
 -6 & -10 & -16 \\
  4 & 7 & 11
\end{pmatrix}
\]
6. Prove that any two non-collinear vectors in \( \mathbb{R}^2 \) are linearly independent.

7. Suppose that \( A \) and \( B \) are square matrices whose rows are orthonormal vectors. Prove that the rows of the matrix \( AB \) are orthonormal vectors.

8. Suppose we have a linear transformation that skews geometrical shapes in the plane vertically so that, e.g. the unit square with its lower left corner at the origin is transformed into a parallelogram with corners located at the points \((0, 0), (1, 1/2), (0, 1), (1, 3/2)\). Determine the matrix representing this linear transformation in the standard basis.

9. Now suppose you are designing graphics software, and you need to be able to take a given image, skew it vertically as described in the previous problem, and then rotate it counter-clockwise through an angle \( \theta \). This can be accomplished all at once by applying a single linear transformation. Determine the matrix representing this transformation in the standard basis.

10. Prove that the following \( n \) vectors are linearly independent in \( \mathbb{R}^n \).

\[
\begin{align*}
\vec{x}_1 &= (1, 1, 1, \ldots, 1) \\
\vec{x}_2 &= (0, 1, 1, \ldots, 1) \\
\vec{x}_3 &= (0, 0, 1, \ldots, 1) \\
& \vdots \\
\vec{x}_n &= (0, 0, 0, \ldots, 1)
\end{align*}
\]